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ON THE EFFECT OF INITIAL STRESSES ON THE OPENING OF A CIRCULAR CRACK*

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Using the theory of small deformations superposed on finite deformation, the problem of loading by uniform pressure on the surfaces of a plane circular crack in and initially extended or compressed (along the crack) elastic medium is considered. The model of incompressible isotropic material of a general form is used. The problem is solved by reduction to a dual integral equation. It is established that the initial stress does not change the order of the singularity of the stress field near the rib crack, but affects the character of stress distribution along the crack rim and the displacements of the crack edges.

The plane problem of cracks of bodies with initial stresses was studied in /1/. The three-dimensional axisymmetric problem of a circular crack was solved in /2/ for a material of a particular form.

1. The equations of equilibrium linearized about the state of a uniform finite deformation of an isotropic incompressible elastic material can be written in the form /3-5/

$$\frac{\partial \theta_{nk}}{\partial x_n}=0, \quad \frac{\partial u_m}{\partial x_m}=0 \tag{1.1}$$

$$\theta_{nk}=s_{nk}-\varepsilon_{nm}t_{mk}-t_{nm}\omega_{mk}$$

$$\varepsilon_{nm}=\frac{1}{2}\left(\frac{\partial u_m}{\partial x_n}+\frac{\partial u_n}{\partial x_m}\right), \quad \omega_{mk}=\frac{1}{2}\left(\frac{\partial u_m}{\partial x_k}-\frac{\partial u_k}{\partial x_m}\right)$$

where x_n are the Cartesian coordinates in the deformed body, u_m are the components of the displacement vector, t_{mk} are the components of stress in the initial deformed state. If the coordinate axes are directed along the principal axes of the stress tensor of the initial state, then for the quantities s_{nk} the representations /4,5/ are valid (the formulas not written out are obtained by circular permutation of subscripts)

$$s_{11}=(\lambda_1^2\Pi_{11}+\lambda_1\Pi_1)\varepsilon_{11}+\lambda_1\lambda_2\Pi_{12}\varepsilon_{22}+\lambda_1\lambda_3\Pi_{13}\varepsilon_{33}+p \tag{1.2}$$

$$s_{12}=s_{21}=\frac{\lambda_1^2+\lambda_2^2}{\lambda_1^2-\lambda_2^2}(\lambda_1\Pi_1-\lambda_2\Pi_2)\varepsilon_{12} \qquad (123)$$

$$\Pi_i=\frac{\partial\Pi}{\partial\lambda_i}, \quad \Pi_{ij}=\frac{\partial^2\Pi}{\partial\lambda_i\partial\lambda_j}, \quad \lambda_1\lambda_2\lambda_3=1$$

where λ_i are the principal stretches in the initial state, Π is the specific potential energy of the material represented as a symmetric function of principal tensions, and p is the function of additional pressure.

Let us consider the unbounded space of incompressible elastic material weakened by an infinitely thin plane circular crack (slit) lying in the horizontal plane $x_3=0$.

We assume that the space is subjected to a finite deformation, produced by a uniform load applied at infinity and acting in the plane of the crack. Evidently with such loading the presence of the crack does not manifest itself, i.e. a uniform deformation in a uniform stress field

$$\lambda_1=\lambda_2=\lambda=\lambda_3^{-1/2}, \quad t_{11}=t_{22}=t=\lambda\Pi_1-\lambda_3\Pi_3 \tag{1.3}$$

$$t_{33}=t_{13}=t_{12}=t_{23}=0$$

corresponds to the stated problem.

On the described finite deformation is superposed a small deformation produced by loading of the crack surface by a uniform pressure of intensity τ. By virtue of the assumption of smallness of the additional deformation the problem is considered in linearized formulation.

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Note that the superposition of solutions of the stated problem and the problem of uniform small deformation in the originally stressed space without crack, which occurs under the action of a uniformly tensile vertical load τ, yields the solution of the problem of opening the crack with unloaded surface by forces applied at infinity.

For axisymmetric deformation in the presence of initial stress-strain state of the form (1.3) the system of equations (1.1) and (1.2) is written as follows:

$$\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)+\nu\left(\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial r\,\partial z}\right)+\frac{\partial q}{\partial r}=0 \qquad (1.4)$$

$$\nu\frac{\partial^2 u}{\partial r\,\partial z}+\varkappa\frac{\partial^2 w}{\partial r^2}+\frac{\nu}{r}\frac{\partial u}{\partial z}+\frac{\varkappa}{r}\frac{\partial w}{\partial r}+\frac{\partial q}{\partial z}=0$$

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0,\qquad q=\theta_{33}=s_{33}$$

$$\mu=2\lambda_3\Pi_3+\lambda^2\Pi_{11}+\lambda_3^2\Pi_{33}-2\lambda\lambda_3\Pi_{13} \qquad (1.5)$$

$$\nu=\lambda_3^2\frac{\lambda_3\Pi_3-\lambda\Pi_1}{\lambda_3^2-\lambda^2},\qquad \varkappa=\lambda^2\frac{\lambda_3\Pi_3-\lambda\Pi_1}{\lambda_3^2-\lambda^2}$$

where r, $z=x_3$ are cylindrical coordinates in the initially deformed body, u and w are components of displacements in the radial and vertical direction, respectively.

The problem of a crack formulated above is equivalent to the boundary value problem for the system of Eqs.(1.4) in the half-space $z>0$ with the following boundary conditions in the plane $z=0$:

$$\theta_{zr}=\nu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)=0,\qquad 0\leqslant r<\infty \qquad (1.6)$$

$$q=-\tau,\ 0\leqslant r<a;\ w=0,\ a\leqslant r<\infty$$

where a is the radius of the circular crack in the initial deformed state.

2. We seek a solution of system (1.4) in the form of integral Hankel expansions

$$u=\int_0^\infty U(z,\alpha)J_1(\alpha r)\,d\alpha,\qquad w=\int_0^\infty W(z,\alpha)J_0(\alpha r)\,d\alpha,\qquad q=\int_0^\infty Q(z,\alpha)J_0(\alpha r)\,d\alpha \qquad (2.1)$$

From Eqs.(1.4) we have

$$U=A_1(\alpha)\,\omega_1E_1+A_2(\alpha)\,\omega_2E_2,\ W=A_1(\alpha)\,E_1+A_2(\alpha)\,E_2 \qquad (2.2)$$

$$Q=\alpha A_1(\alpha)\,\omega_1(\nu-\mu+\nu\omega_1^2)\,E_1+\alpha A_2(\alpha)\,\omega_2(\nu-\mu+\nu\omega_2^2)\,E_2$$

$$(E_i=\exp(-\alpha\omega_iz),\ i=1,\,2)$$

where ω_1, ω_2 are the roots of equation

$$\nu\omega^4-(\mu-2\nu)\,\omega^2+\varkappa=0 \qquad (2.3)$$

which have a positive real part.

Taking into account that $\varkappa\nu^{-1}=\lambda^2\lambda_3^{-2}>0$, the solution of Eq.(2.3) can be represented in the Descartes-Euler form

$$2\omega=\pm\sqrt{\Delta_+}\pm\sqrt{\Delta_-},\ \Delta_\pm=\mu\nu^{-1}-2\pm2\sqrt{\varkappa\nu^{-1}} \qquad (2.4)$$

From (2.4) follows that the necessary and sufficient condition of existence of two roots with positive real parts is the inequality

$$\Delta_+>0 \qquad (2.5)$$

When condition (2.5) is violated, all roots of Eq.(2.3) are pure imaginary, i.e. the boundary value problem (1.4), (1.6) has no solutions that attenuate as $z\to\infty$.

Referring to (1.5), we see that conditions (2.5) are satisfied for the following constraint on the function of the elastic potential Π:

$$\frac{\lambda\Pi_1-\lambda_3\Pi_3}{\lambda-\lambda_3}>0 \qquad (2.6)$$

$$\lambda^2\Pi_{11}+\lambda_3^2\Pi_{33}-2\lambda\lambda_3\Pi_{13}+2\lambda\lambda_3\frac{\Pi_1+\Pi_3}{\lambda+\lambda_3}>0$$

Relations (2.6) differ only slightly (the sign $>$ instead of $\geqslant$) of constraint on the elastic potential of incompressible material, which are included in the number of necessary and sufficient conditions of fulfilment of Hadamard inequalities /6/, the criterion of reality of plane wave velocities in an initially stressed field.

From the first boundary condition of (1.6) we obtain

$$A_2 = -(1+\omega_1^2)(1+\omega_2^2)^{-1}A_1 \tag{2.7}$$

According to (2.2) and (2.7), we have

$$u(r,z) = \int_0^\infty A_1(\alpha)\left(\omega_1 E_1 - \frac{1+\omega_1^2}{1+\omega_2^2}\omega_2 E_2\right) J_1(\alpha r)\,d\alpha \tag{2.8}$$

$$w(r,z) = \int_0^\infty A_1(\alpha)\left(E_1 - \frac{1+\omega_1^2}{1+\omega_2^2}E_2\right) J_0(\alpha r)\,d\alpha$$

$$q(r,z) = \int_0^\infty A_1(\alpha)\,\alpha\,[\omega_1(\nu-\mu+\nu\omega_1^2)E_1 -$$
$$\omega_2(\nu-\mu+\nu\omega_2^2)(1+\omega_1^2)(1+\omega_2^2)^{-1}E_2]\,J_0(\alpha r)\,d\alpha$$

Satisfying the boundary conditions (1.6), we obtain for the function $A_1(\alpha)$ the dual integral equation

$$\int_0^\infty A_1(\alpha)\,\alpha J_0(\alpha r)\,d\alpha = -\Phi(\lambda)\tau, \quad 0 \leqslant r < a \tag{2.9}$$

$$\int_0^\infty A_1(\alpha)\,J_0(\alpha r)\,d\alpha = 0, \quad a < r < \infty$$

$$\Phi(\lambda) = (1+\omega_2^2)(\omega_1-\omega_2)^{-1}\varphi(\lambda)$$
$$\varphi(\lambda) = [(1+\omega_1^2)(1+\omega_2^2)\nu + (\omega_1\omega_2 - 1)\mu]^{-1}$$

3. Equation (2.9) is in the class of equations whose solution is of the form /7/

$$A_1(\alpha) = \frac{2a^2}{\pi}\Phi(\lambda)\tau\left(\frac{\cos\alpha a}{\alpha a} - \frac{\sin\alpha a}{\alpha^2 a^2}\right) \tag{3.1}$$

Using /8/, from (2.9) and (3.1) we obtain

$$u(r,0) = {}^1/_2\tau(\omega_1\omega_2 - 1)\,r\varphi(\lambda) \quad (r \leqslant a) \tag{3.2}$$
$$w(r,0+) = 2\tau\pi^{-1}(\omega_1+\omega_2)\sqrt{a^2-r^2}\,\varphi(\lambda) \quad (r \leqslant a)$$
$$s_{33}(r,0) = \frac{2\tau}{\pi}\left(\frac{a}{\sqrt{r^2-a^2}} - \arcsin\frac{a}{r}\right) \quad (r > a)$$

Formulas (3.2) show that for any incompressible isotropic material the order of the singularity of stresses near the crack tip is the same as in the problem of a circular crack without allowance for initial stresses /9,10/.

As an example, let us consider the case of neo-Hookian material, for which the elastic potential is specified by the formula

$$\Pi = {}^1/_2 G(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) \tag{3.3}$$

where G is the shear modulus. From (1.5) and (2.4) we have

$$\nu = G\lambda^{-4}, \ \mu = G(\lambda^2 + 3\lambda^{-4}), \ \varkappa = G\lambda^2, \ \omega_1 = 1, \ \omega_2 = \lambda^3 \tag{3.4}$$

For the displacements of the upper edge of the crack from (3.2) and (3.4) we obtain

$$u = \frac{\tau(\lambda^3-1)N(\lambda)r}{4G(1+\lambda^3)}, \quad w = \frac{\tau}{4G}N(\lambda)\sqrt{a^2-r^2}, \ N(\lambda) = \frac{2\lambda^4(1+\lambda^3)}{\lambda^9+\lambda^6+3\lambda^3-1} \tag{3.5}$$

Formulas (2.8) for a neo-Hookian material reduce to the form

$$u(r,z) = \int_0^\infty A_1(\alpha)\left(e^{-\alpha z} - \frac{2\lambda^3}{1+\lambda^6}e^{-\alpha\lambda^3 z}\right)J_1(\alpha r)\,d\alpha \tag{3.6}$$

$$w(r,z) = \int_0^\infty A_1(\alpha)\left(e^{-\alpha z} - \frac{2}{1+\lambda^6}e^{-\alpha\lambda^3 z}\right)J_0(\alpha r)\,d\alpha$$

$$s_{33}(r,z) = G\int_0^\infty A_1(\alpha)\,\alpha\left[-\frac{1+\lambda^6}{\lambda^4}e^{-\alpha z} + \frac{4e^{-\alpha\lambda^3 z}}{\lambda(1+\lambda^6)}\right]J_0(\alpha r)\,d\alpha$$

$$A_1(\alpha) = \frac{a^2\tau}{\pi G}\,\frac{1+\lambda^6}{1-\lambda^6}N(\lambda)\left(\frac{\cos\alpha a}{\alpha a} - \frac{\sin\alpha a}{\alpha^2 a^2}\right)$$

When $\lambda \to 1$, i.e. when initial stresses are removed, formulas (3.5) become the solution of the problem of a crack in an unstressed body /9,10/ (for the case of Poisson's ratio equal to 0.5, since the material is incompressible).

The relations (3.5) show that the presence of initial stresses in the body results in the appearance of radial displacements on the surface of the crack extended by a uniform pressure. Moreover, as shown by (3.6), the initial stresses affect the character of distribution of stresses and displacements around the crack edge.

The coefficient $N(\lambda)$ appearing in expressions for displacements (3.5) in the interval $\lambda^* < \lambda < \infty$, where $\lambda^* \approx 0.667$, is monotonically decreasing; and $N(\lambda) \to \infty$ as $\lambda \to \lambda^*$. When the magnitude of initial compression approaches λ^*, the displacements of the crack edges increase to the infinity. This means that for $\lambda \leqslant \lambda^*$ the uniform stress-strain state of the compressed body with a crack is unstable. Note that the critical value of the initial compression λ^* coincides with that of λ, at which the instability of the compressed half-space /11/ is initiated.

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